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Unsteady flow at a stagnation point

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The flow at an axisymmetric stagnation point is considered when the outer, inviscid flow is oscillatory with zero mean. It is shown that following the commencement of the flow at an initial instant there is a breakdown of the solution, after a finite time, as fluid erupts from the boundary at the stagnation point. This is interpreted as the origin of the jet-like flow observed in the experiments of Amin & Riley (1990). It is demonstrated that an in-plane, torsional motion of the boundary, either steady or oscillatory, can suppress the singular behaviour.

1. Introduction

The present work has been motivated by a previous investigation carried out by Amin & Riley (1990), in which the steady streaming from a sphere due to a pulsating source was analysed. A limiting case of this is when the source retreats to infinity, but its strength becomes unbounded. The analogue of that case is when the sphere performs unidirectional oscillations in a fluid otherwise at rest. In a simple experiment, Amin & Riley were able to demonstrate one of the main features of the flow, namely the appearance of a quasi-steady jet-like flow along the axis of oscillation. This they visualized by introducing dye into the fluid. Their figure 7, reproduced here as figure 1, clearly shows the phenomenon. In the experiment the sphere was coated with dye before being set into motion. The dye was seen to move towards the stagnation points before erupting from the surface to form the jets shown in figure 1.

In this paper we analyse the eruption phenomenon, and argue that a torsional motion of the sphere can prevent it. Of course, if a sphere performs only a steady rotation about an axis then a different type of jet emerges from it, namely an equatorial jet in a plane perpendicular to the axis of rotation, as observed by Bowden & Lord (1963). For such a motion, started from rest, the jet formation is by an eruption from the equator, as has been examined by Banks & Zaturska (1979). Such eruptions manifest themselves as singularities, at a finite time, in the solution of the governing equations which is localized to the point or line on which the eruption takes place; these are often conceived as boundary-layer collisions. In the present investigation we examine the phenomenon described above by studying the unsteady flow in the neighbourhood of the stagnation point when the outer inviscid flow performs harmonic oscillations with zero mean. Our approach is similar to that adopted by Riley & Vasantha (1989), who examined such a flow at a two-dimensional stagnation point. For arbitrary values of a suitably defined frequency parameter a direct integration of the governing equations, following the initiation of the motion, leads to an estimate of the breakdown time. This is refined by a knowledge of the structure of the singularity associated with the breakdown. For the two-dimensional case, Riley & Vasantha showed that the singularity was that of Banks & Zaturska, associated with the notion of the boundary layers colliding along a line. The present situation is different in the

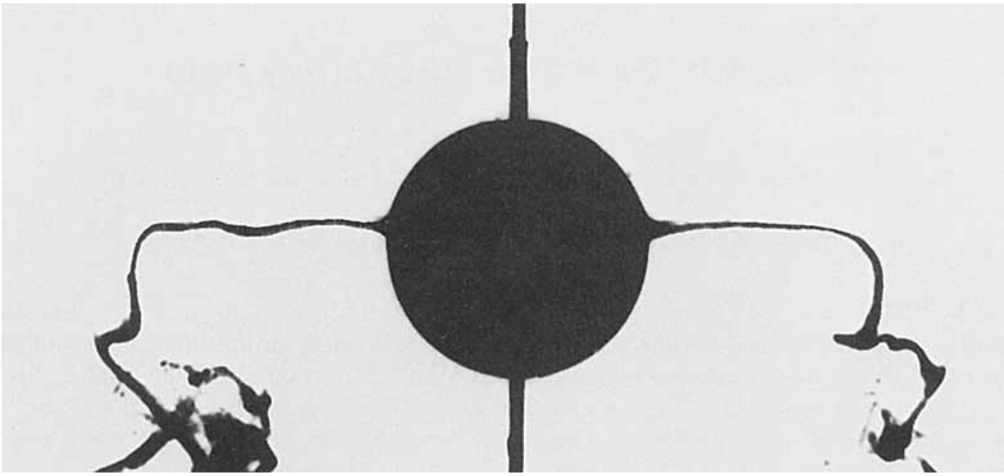

FIGURE 1. A visualization of the jets issuing from the sphere, along the oscillation axis, in the experiment of Amin & Riley (1990).

sense that the boundary layer is converging onto a point, rather than a line, in the eruptive process. And in this case the singular behaviour is basically that described by Brown & Simpson (1982) for the eruption of a free-convective plume from the upper pole of an impulsively heated sphere. The calculations are supplemented by an analysis appropriate to large and small values of the frequency parameter. In particular the high-frequency case (which actually corresponds to small values of the frequency parameter) clearly demonstrates the basic features of the flow in question, in the sense that the time-averaged flow is seen to lead to a radial inflow towards the stagnation point, from which fluid must obviously then be ejected from the boundary. The singular behaviour that we report here may be contrasted with that at a rear stagnation point for impulsive steady flow. This has been considered, in the axisymmetric case, by Howarth (1973). In that situation there is no breakdown of the solution at a finite time, and the boundary layer thickens indefinitely. In a final section of the paper we show that a torsional movement of the boundary at the stagnation point can suppress the eruptive singular behaviour described above. This we do by considering two particular cases, namely that in which the boundary rotates steadily, and secondly the case where it performs torsional oscillations. It is the net radial motion outwards, which is induced by the torsional motion of the boundary, overcoming the radial inflow referred to above that suppresses the eruption and the flow assumes a quasi-steady state.

2. Governing equations

We take the origin at the stagnation point, (r, θ, z) are cylindrical polar coordinates with (u, v, w) the corresponding velocity components: the boundary lies in the plane $z = 0$. With axial symmetry assumed the governing equations for unsteady flow are, in the boundary-layer limit,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial r}+\nu\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\frac{\partial^2 v}{\partial z^2}, \tag{2}$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw) = 0, \tag{3}$$

where t is time, ν is the kinematic viscosity, and U_1 is the radial component of velocity outside the boundary layer; the corresponding azimuthal component is zero. We now suppose that outside the boundary layer the radial and axial components of velocity (U_1, W_1) are given by

$$U_1 = r\Omega_1 \cos\omega_1 t, \quad W_1 = -2\Omega_1 z \cos\omega_1 t, \tag{4}$$

with, as already mentioned, no azimuthal velocity. However, we do allow the possibility of azimuthal motion of the boundary, $z = 0$, itself, such that

$$v(r,0,t) = r\Omega_2\,\phi(\tau), \quad \text{where} \quad \tau = \omega_1 t. \tag{5}$$

The quantities ω_1, Ω_1 and Ω_2 are all constants with dimension $(\text{time})^{-1}$.

Following the classical reduction of the boundary-layer equations for axisymmetric stagnation-point flow, and allowing for an azimuthal motion, we now write

$$u = r\Omega_1 F'(\eta,\tau), \quad v = r\Omega_2 G(\eta,\tau), \quad w = -2\Omega_1\left(\frac{2\nu}{\omega_1}\right)^{\frac{1}{2}} F(\eta,\tau), \tag{6a–c}$$

where

$$\eta = \left(\frac{\omega_1}{2\nu}\right)^{\frac{1}{2}} z \quad \text{and} \quad \tau = \omega_1 t, \tag{7}$$

and we have used a prime to denote differentiation with respect to η. With (3) satisfied identically, equations for F and G are derived by substituting (4), (6) and (7) into (1) and (2) to give

$$\frac{\partial F'}{\partial \tau}+\lambda_1(F'^2-2FF'')-\lambda_2 G^2 = -\sin\tau+\lambda_1\cos^2\tau+\tfrac{1}{2}F''', \tag{8}$$

$$\frac{\partial G}{\partial \tau}+2\lambda_1(F'G-FG') = \tfrac{1}{2}G'', \tag{9}$$

where

$$\lambda_1 = \Omega_1/\omega_1, \quad \lambda_2 = \Omega_2/\omega_1. \tag{10}$$

The boundary conditions for (8) and (9) are

$$F(0,\tau) = F'(0,\tau) = 0, \quad G(0,\tau) = \phi(\tau), \quad \text{all } \tau, \tag{11}$$

$$F'(\infty,\tau) = \cos\tau, \quad G(\infty,\tau) = 0, \quad \text{all } \tau, \tag{12}$$

where ϕ is, as yet, unspecified. For all the cases considered, we assume that the motion is started impulsively at $\tau = 0$, so that conditions (11) and (12) are supplemented by

$$F'(\eta,0+) = 1, \quad G(\eta,0+) = 0, \quad \text{for all } \eta. \tag{13}$$

In the next section we concentrate on the situation in which there is no azimuthal motion, so that $\phi = 0$ for all τ, and we demonstrate how this unsteady stagnation-point flow breaks down in a finite time. In §4 we show how an azimuthal motion of the boundary can suppress this singular behaviour.

3. The case $\phi(\tau) = 0$

In relation to the problem studied by Amin & Riley (1990), this case may be interpreted as corresponding to the flow in the neighbourhood of the forward stagnation point on a sphere which is set into oscillatory motion at $\tau = 0$.

For convenience we now write (8) as the coupled equations

$$\frac{\partial H}{\partial \tau}+\lambda_1(H^2-2FH') = -\sin\tau+\lambda_1\cos^2\tau+\tfrac{1}{2}H'', \quad F' = H, \tag{14a, b}$$

which are to be solved subject to conditions (11)–(13). The numerical integration of (14) is straightforward, and is carried out iteratively following a quasi-linearization of the quadratic term in H of equation (14a). The solution for H is advanced in time using a Crank–Nicolson method, whilst (14b) is solved for F using simple quadrature. Thus at each time step there is iteration between (14a) and (14b) until convergence is achieved. As $\tau \to 0, H \sim \text{erf}\{\eta/(2\tau)^{\frac{1}{2}}\}$, as a consequence of which we commence our numerical integration of (14) in variables $(\bar{\eta}, \tau)$ where $\bar{\eta} = \eta/\tau^{\frac{1}{2}}$, reverting to variables (η, τ) at $\tau = 1$, when $\bar{\eta} = \eta$. The solution is advanced by increments $\delta\tau = 0.01$, with a spatial mesh size $\delta\bar{\eta} = \delta\eta = 0.1$. The outer boundary has been set at $\bar{\eta}_\infty = \eta_\infty = 500$.

The solution of (14a, b) depends upon the single parameter λ_1, and in the range $0.65 \leqslant \lambda_1 \leqslant 6.5$ for which we have carried out a numerical investigation, as described above, the solution breaks down at a finite time τ_s. This breakdown manifests itself as an eruption of fluid from the boundary layer in which the axial velocity becomes unbounded, as does the maximum radial *inflow* to feed this eruption. The eruption has in common with others that have been studied, as detailed in §1, an inviscid nature. A balance of the first two terms of (14a) suggests that we may expect the maximum value of $|H|$ to vary as $(\tau_s-\tau)^{-1}$, and this is indeed consistent with our numerical results. The basic structure of the singularity implied by this result is identical with that considered by Brown & Simpson (1982) for the erupting flow at the upper pole of an impulsively heated sphere, and is not reproduced here, except to note that F varies as $(\tau_s-\tau)^{-\frac{5}{2}}$ as $\tau \to \tau_s$, and the inviscid region in which the singularity is centred increases in thickness like $(\tau_s-\tau)^{-\frac{3}{2}}$. The inviscid region is flanked by regions in which viscous effects are important. For our purpose the important features of the singular structure are that, as $\tau \to \tau_s$,

$$\max|H| \sim \mu_1(\tau_s-\tau)^{-1}, \tag{15a}$$

$$F_d = \lim_{\eta\to\infty}(\eta\cos\tau-F) \sim \mu_2(\tau_s-\tau)^{-\frac{5}{2}}, \tag{15b}$$

where μ_1 and μ_2 are constants. The quantity F_d in (15b) is, from (6c), seen to be a measure of the viscous displacement velocity. To determine the breakdown time τ_s we proceed as follows. With the outer edge of the boundary layer fixed at $\eta = \eta_\infty$ we monitor the quantity $|\partial H/\partial\eta|$ at $\eta = \eta_\infty - 5\delta\eta$. When this exceeds 0.1 the calculation is terminated to give an approximate value τ_a of the breakdown time τ_s. This may now be improved, by use of (15), as follows. We assume linear relationships among $F_d^{-\frac{2}{5}}$, $(\max|H|)^{-1}$ and τ, and then use successive pairs of points from $\tau = \tau_a - 10\delta\tau$ to linearly extrapolate these quantities to zero. In this way we have a succession of estimates of τ_s, converging to our final choice. The final estimates of τ_s from each of (15a, b) agree to within less than 0.2%, and we are confident that the results we show in figure 2, from the numerical solution of (14), are accurate to at least two decimal places. At this point we remark that the eruptive behaviour of the solution is independent of the manner in which the flow is initiated, impulsively for the results described above. For example, we have repeated the calculations with $F'(\infty, \tau) = \sin\tau$, $(1-e^{-\tau})\sin\tau$, $(1-e^{-\tau^2})\sin\tau$ corresponding, respectively, to $F' \sim \tau, \tau^2, \tau^3$ as $\tau \to 0$. The same behaviour as described above is observed, with a delay in the onset of singular behaviour of approximately $\frac{1}{2}\pi$ for the larger values of λ_1.

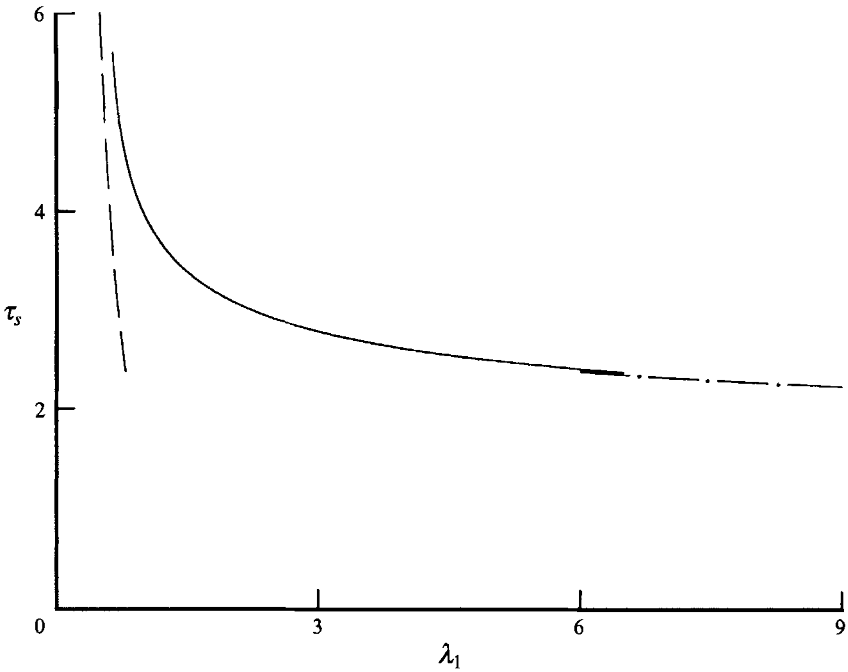

FIGURE 2. The breakdown time τ_s of the solution as a function of λ_1: ——, from the numerical solution of equation (14); - - - -, from the asymptotic result (34) for $\lambda_1 \ll 1$; —·—·—, from the asymptotic result (24) for $\lambda_1 \gg 1$.

The results we have obtained in the manner described strongly suggest that as $\lambda_1 \to \infty$, $\tau_s \to$ const., and as $\lambda_1 \to 0$, $\tau_s \to \infty$; we now investigate each of these possibilities in turn.

(i) $\lambda_1 \gg 1$

With $\omega_1 \ll \Omega_1$, we interpret this as a low-frequency situation in which the solution is, over an initial period of time, the quasi-steady stagnation-point flow. To acknowledge this formally we write

$$F(\eta,\tau) = (\cos\tau)^{\frac{1}{2}}\lambda_1^{-\frac{1}{2}}\bar{F}(\zeta), \quad \text{where} \quad \zeta = (\lambda_1 \cos\tau)^{\frac{1}{2}}\eta, \tag{16}$$

where, from (8), $\bar{F}(\zeta)$ satisfies

$$\tfrac{1}{2}\bar{F}''' + 2\bar{F}\bar{F}'' - \bar{F}'^2 + 1 = O(\sin\tau/\lambda_1\cos^2\tau), \tag{17}$$

with
$$\bar{F}(0) = \bar{F}'(0) = 0, \quad \bar{F}'(\infty) = 1, \tag{18}$$

and a prime now denotes differentiation with respect to ζ. We see from (17) and (18) that $\bar{F}(\zeta)$ is the classical, steady stagnation-point solution which will fail, as the right-hand side of (17) shows, as $\tau \to \frac{1}{2}\pi$. Specifically, this form of the solution will be inadequate when $\lambda_1^{\frac{1}{2}}\cos\tau = O(1)$. In that case the time variation of F cannot be ignored, which leads to the introduction of new variables, with

$$\tau = \tfrac{1}{2}\pi - \lambda_1^{-\frac{1}{2}}T, \quad \bar{\zeta} = \lambda_1^{\frac{1}{4}}\eta, \quad F(\eta,\tau) = \lambda_1^{-\frac{3}{4}}\mathscr{F}(\bar{\zeta},T), \tag{19}$$

where, from (8), $\mathscr{F}$ satisfies

$$\frac{\partial\mathscr{F}'}{\partial T} + \tfrac{1}{2}\mathscr{F}''' + 2\mathscr{F}\mathscr{F}'' - \mathscr{F}'^2 + T^2 - 1 = 0, \tag{20}$$

with $$\mathscr{F}(0,T)=\mathscr{F}'(0,T)=0, \quad \mathscr{F}'(\infty,T)=T, \tag{21}$$

and a prime in (17), (18) denotes differentiation with respect to $\bar{\zeta}$. In order to solve (20) subject to (21) we need a further condition, which expresses the fact that the solution of (20) matches, as $T\to\infty$, the quasi-steady stagnation-point solution $\bar{F}(\zeta)$. To allow this we write

$$\mathscr{F}(\bar{\zeta},T)=(T/T_\infty)^{\frac{1}{2}}\bar{\mathscr{F}}(\xi), \quad \text{where} \quad \xi=(T/T_\infty)^{\frac{1}{2}}\bar{\zeta}, \tag{22}$$

and $\bar{\mathscr{F}}$ satisfies, denoting differentiation with respect to ξ by a prime,

$$\tfrac{1}{2}\bar{\mathscr{F}}'''+2\bar{\mathscr{F}}\bar{\mathscr{F}}''-\bar{\mathscr{F}}'^2+T_\infty^2=0, \tag{23a}$$

with $$\bar{\mathscr{F}}(0)=\bar{\mathscr{F}}'(0)=0, \quad \bar{\mathscr{F}}'(\infty)=T_\infty. \tag{23b}$$

In deriving (23) we have neglected terms $O(T_\infty^2/T^2, T_\infty/T, T_\infty/T^2)$, when T is large, compared to terms $O(T_\infty^2)$. The solution of (23) is, of course, simply a scaled version of the classical stagnation-point flow, and we use it as a condition at $T=T_\infty$ to commence the integration of (20), (21), with T decreasing. We take $T_\infty=10$, and the strategy for integrating (20) that we have adopted mirrors that of §2, by splitting the equation into two as in (14). As with that equation a breakdown of the solution occurs at $T=T_s$ where $T_s=-2.01$, so that we have, from (19),

$$\tau_s \sim \tfrac{1}{2}\pi+\frac{2.01}{\lambda_1^{\frac{1}{2}}} \quad \text{as} \quad \lambda_1\to\infty. \tag{24}$$

This result is included in figure 2, and shows remarkable agreement with the results we have already obtained by the methods of §2, for $\lambda_1>6$. We now consider the other limiting case.

(ii) $\lambda_1 \ll 1$

We seek a solution of (8) in this case by writing

$$F(\eta,\tau)=F_0(\eta,\tau)+\lambda_1\{F_1^{(s)}(\eta)+F_1^{(u)}(\eta,\tau)\}+O(\lambda_1^2), \tag{25}$$

where we have adopted the superscript notation 's' to denote quantities independent of τ, and 'u' for quantities that are time-dependent. The reason for splitting the terms $O(\lambda_1)$ in (25) in this manner will become apparent below. At $O(1)$ we have, from (8)–(13), the solution

$$F_0(\eta,\tau)=\left\{\eta+\frac{1-\mathrm{i}}{2}(\mathrm{e}^{-(1+\mathrm{i})\eta}-1)\right\}\mathrm{e}^{\mathrm{i}\tau}, \tag{26}$$

where the real part of (26) is to be understood. We note that transients associated with the initial condition have been ignored; these decay as $\tau^{-\frac{5}{2}}$ and are of no concern to us in what follows. At $O(\lambda_1)$ interaction between the nonlinear terms yields a contribution that is independent of τ so that $F_1^{(s)}$ satisfies

$$F_1^{(s)'''}=2(F_0'^2-2F_0F_0'')^{(s)}-1, \tag{27a}$$

with $$F_1^{(s)}(0)=F_1^{(s)'}(0)=0. \tag{27b}$$

It is well known, see for example Stuart (1963), that in these steady-streaming situations it is not possible to satisfy the condition $F_1^{(s)'}(\infty)=0$ within the Stokes boundary layer of thickness $O(\nu/\omega)^{\frac{1}{2}}$ that we are working. The solution of (27) which has $F_1^{(s)'}$ bounded as $\eta\to\infty$ is

$$F_1^{(s)}(\eta)=-\eta\,\mathrm{e}^{-\eta}\sin\eta-\tfrac{1}{2}\mathrm{e}^{-\eta}(5\cos\eta+3\sin\eta)-\tfrac{1}{8}\mathrm{e}^{-2\eta}-\tfrac{5}{4}\eta+\tfrac{21}{8}, \tag{28}$$

so that
$$F'(\eta,\tau) \sim \cos\tau - \tfrac{5}{4}\lambda_1 + O(\lambda_1^2) \quad \text{as} \quad \eta \to \infty. \tag{29}$$

We see, from (29), that in order to match our solution with (4), it is necessary to introduce an outer boundary layer in which the adjustment is made. Furthermore (29) demonstrates, more clearly than we have seen previously, the origins of the eruptive failure of the solution of our governing equations. Thus, a time average of (29) shows that there is a net radial inflow to the stagnation point which inevitably leads to an eruption from it. To accommodate the time-independent element of the flow that has emerged at $O(\lambda_1)$ we introduce variables appropriate to an outer region as

$$F(\eta,\tau) = \eta\cos\tau + f(\bar{\eta},\bar{\tau}), \quad \text{where} \quad \bar{\eta} = \lambda_1\eta, \quad \bar{\tau} = \lambda_1^2\tau. \tag{30}$$

In (30) the scaling of the spatial coordinate allows us to match the solution in the outer region with (29), and our choice of timescale is motivated as follows. From (6*a*) and (25) we see that the time-averaged radial velocity has magnitude $O(r\Omega_1^2/\omega_1)$ which suggests that an appropriate timescale on which the eruption will take place is ω_1/Ω_1^2 or $\lambda_1^{-2}\omega_1^{-1}$ as implied by (30).

If the variables (30) are introduced into (8), and the resulting equation is averaged on the scale of τ, then f satisfies

$$\frac{\partial f'}{\partial \bar{\tau}} - \tfrac{1}{2}f''' - 2ff'' + f'^2 = 0, \tag{31}$$

where a prime now denotes differentiation with respect to $\bar{\eta}$, together with

$$f(0,\bar{\tau}) = 0, \quad f'(0,\bar{\tau}) = -\tfrac{5}{4}, \quad f'(\infty,\bar{\tau}) = 0, \quad \bar{\tau} > 0; \tag{32}$$

and
$$f'(\bar{\eta},0+) = 0, \quad \bar{\eta} > 0. \tag{33}$$

We note that the condition (33) at $\bar{\tau} = 0$ conceals an uncertainty in the origin of time, for this condition, on the scale of τ. Our strategy for the solution of (31)–(33) is exactly that set out in §2, with the initial distribution of f' now given as

$$f'(\bar{\eta},\bar{\tau}) = -\tfrac{5}{4}[1 - \operatorname{erf}\{\bar{\eta}/(2\bar{\tau})^{\frac{1}{2}}\}].$$

As anticipated the solution fails, in the eruptive manner that we have encountered earlier, at $\bar{\tau}_s = 1.524$ so that

$$\tau_s \sim \frac{1.524}{\lambda_1^2} \quad \text{as} \quad \lambda_1 \to 0. \tag{34}$$

This result is also shown in figure 2 where it is seen to be consistent with our numerical solutions of (14).

This concludes our discussion of the development of a singularity in the oscillatory flow at the stagnation point when the boundary is at rest.

4. The case $\phi(\tau) \neq 0$

As we have noted in the previous section, it is the time-averaged non-zero radial inflow at the stagnation point that initiates the eruptive breakdown. As is well known any rotation, be it steady or unsteady, of the boundary about the *z*-axis will result in a net radial outflow of fluid. If this overwhelms the induced flow, then there will be no breakdown of the solution. In this section we restrict our attention to two such situations. In the first the plane boundary is set into a uniform rotation, whilst in the second it performs an oscillatory motion.

(a) $\lambda_1 \ll 1$

(i) $\phi(\tau) \equiv 1, \tau > 0$. For $\lambda_1 \ll 1$ we expand $G(\eta, \tau)$ in a similar manner to F in (25). Thus we write

$$G(\eta,\tau) = G_0(\eta,\tau) + \lambda_1\{G_1^{(s)}(\eta) + G_1^{(u)}(\eta,\tau)\} + O(\lambda_1^2) \tag{35}$$

so that G_0 satisfies
$$G_0'' - \frac{\partial G_0}{\partial \tau} = 0, \tag{36a}$$

with $G_0(0,\tau) = 1, \quad G_0(\infty,\tau) = 0 \quad \text{for} \quad \tau > 0; \qquad G_0(\eta, 0) = 0 \quad \text{for} \quad \eta > 0.$ (36b)

The solution of (36) for the Stokes layer is simply $G_0 \equiv 1$, ignoring transients, which indicates that the rotation of the boundary is impressed upon the Stokes layer. Now, it is clear that for a sufficiently large rotation rate, Ω_2, the radial inflow can always be overcome. One might ask, therefore, how small a rotation rate is required such that the outflow and inflow are comparable? In order for the centrifugal term, G^2, in (8) to be comparable with the other terms on the left-hand side of that equation in the outer boundary layer, which we have already defined by (30), we require $\lambda_2 = O(\lambda_1^3)$. So, let

$$\lambda_2 = \beta\lambda_1^3. \tag{37}$$

With λ_2 so defined, the equation for $F_1^{(s)}$, (27), is unchanged with solution (28). The radial inflow still persists at the edge of the Stokes layer, and in the boundary layer beyond we again introduce the variables (30) together with

$$G(\eta,\tau) = g(\bar{\eta},\bar{\tau}). \tag{38}$$

And then from (8), (9), averaging again on the scale of τ, the 'fast' timescale, the equations satisfied by f and g are

$$\frac{\partial f'}{\partial \bar{\tau}} + f'^2 - 2ff'' - \beta g^2 = \tfrac{1}{2}f''', \tag{39a}$$

$$\frac{\partial g}{\partial \bar{\tau}} + 2(f'g - fg') = \tfrac{1}{2}g'', \tag{39b}$$

and we note that setting $\beta = 0$ in (39a) recovers (31). The boundary conditions for (39) are

$$f(0,\bar{\tau}) = 0, \quad f'(0,\bar{\tau}) = -\tfrac{5}{4}, \quad g(0,\bar{\tau}) = 1, \quad f'(\infty,\bar{\tau}) = g(\infty,\bar{\tau}) = 0, \quad \bar{\tau} > 0; \tag{40}$$

with
$$f'(\eta, 0+) = g(\eta, 0+) = 0, \quad \eta > 0. \tag{41}$$

Now, there will be a critical value of β, say β_c, such that for $\beta < \beta_c$ the developing solution of (39) will break down, in the manner that we have described in §3, whilst for $\beta > \beta_c$ the solution will asymptote to a steady state, with a net radial outflow. To find β_c, we proceed as follows. For $\beta < \beta_c$, since the solution fails at a finite value of $\bar{\tau}$, we may anticipate that the steady-state form of (39), that is with the $\bar{\tau}$-derivative terms ignored, will have no solution. By contrast for $\beta > \beta_c$ we expect them to yield the steady-state solution as anticipated above. Our procedure then is to find, for a sufficiently large value of β, the steady-state solution of (39) by a direct integration of the steady-state equations, and then to decrease β by small amounts, $\delta\beta$ say, using the previously converged solution as the initial estimate in the iterative solution at the new value of β. We find that for β sufficiently small, the solution of the steady-state equations fails to converge, which we interpret as non-existence of the solution, for $\beta < \beta_c$. The estimate of β_c itself can be refined by reducing the increment $\delta\beta$. We find that

$$\beta_c = 1.149. \tag{42}$$

Before turning to our second case, we note from (37) that, since $\lambda_1 \ll 1$, the steady rotation rate that is required to inhibit the singular behaviour is very small indeed.

(ii) $\phi(\tau) = \cos(\tau+\alpha)$, $\tau > 0$. In this case the plane boundary performs torsional oscillations with the same frequency as the outer flow, but with phase difference α.

We again expand F, G as in (25) and (35). The solution for F_0 in (26) is unchanged; G_0 satisfies (36) except that the condition at $\eta = 0$ is replaced by $G_0(0,\tau) = \cos(\tau+\alpha)$, so that, if we again ignore the initial transients,

$$G_0(\eta,\tau) = \mathrm{e}^{-\eta+\mathrm{i}(\tau+\alpha-\eta)}. \tag{43}$$

In contrast to case (i), where the azimuthal component of velocity was independent of τ at $O(1)$, we see immediately from the interactions in (9) that the time-averaged azimuthal velocity will now be $O(\lambda_1)$. Further, from (8), we recognize that the azimuthal motion will affect the time-averaged radial motion at $O(\lambda_2)$. This leads us to set

$$\lambda_2 = \beta\lambda_1. \tag{44}$$

The equation (27) for $F_1^{(s)}$ now includes a term $-\beta G_0^{2(s)}$ on its right-hand side, which in turn modifies the solution which becomes

$$F_1^{(s)}(\eta) = -\eta\,\mathrm{e}^{-\eta}\sin\eta - \tfrac{1}{2}\mathrm{e}^{-\eta}(5\cos\eta+3\sin\eta) - \tfrac{1}{8}(1-\beta)\,\mathrm{e}^{-2\eta} - \tfrac{1}{4}(5-\beta)\,\eta + \tfrac{1}{8}(21-\beta). \tag{45}$$

The equation for $G_1^{(s)}$ is $$G_1^{(s)\prime\prime} = 4(F_0' G_0 - F_0 G_0')^{(s)}, \tag{46}$$

with solution satisfying $G_1^{(s)}(0) = 0$, and bounded at infinity,

$$G_1^{(s)}(\eta) = \eta\,\mathrm{e}^{-\eta}\{\cos(\eta-\alpha) - \sin(\eta-\alpha)\} + 2\,\mathrm{e}^{-2\eta}(\sin\alpha - \cos\alpha) + 2\,\mathrm{e}^{-\eta}\{\cos(\eta-\alpha) - \sin(\eta-\alpha)\}. \tag{47}$$

We now have a situation in which a steady streaming persists at the edge of the Stokes layer in both the radial and azimuthal directions. For, as $\eta\to\infty$, we have

$$F' \sim \cos\tau + \tfrac{1}{4}\lambda_1(\beta-5) + O(\lambda_1^2), \tag{48a}$$

$$G \sim \tfrac{1}{2}\lambda_1(3\cos\alpha - 7\sin\alpha) + O(\lambda_1^2). \tag{48b}$$

This leads us to consider the flow in the outer boundary layer for which we again introduce the variables defined by (30) and (38) with f, g again satisfying (39), but with the boundary conditions replaced by

$$f(0,\bar{\tau}) = 0, \quad f'(0,\bar{\tau}) = \tfrac{1}{4}(\beta-5), \quad g(0,\bar{\tau}) = \tfrac{1}{2}(3\cos\alpha - 7\sin\alpha), \tag{49a}$$

$$f'(\infty,\bar{\tau}) = g(\infty,\bar{\tau}) = 0. \tag{49b}$$

As in case (i) we argue that there will be some critical value β_c, of β, such that for $\beta > \beta_c$ steady-state solutions of (39) will exist, but that for $\beta < \beta_c$ the solution will break down at a finite time. The difference now, as we see from (49), is that $\beta_c = \beta_c(\alpha)$. One point we note immediately is that, regardless of the value of α, if $\beta > 5$ we have $f'(0,\bar{\tau}) > 0$ and there will be no breakdown of the solution. To find β_c we fix the value of α, with $0 \leqslant \alpha \leqslant \pi$, and proceed exactly as in case (i). The results of this investigation are shown in figure 3, and we note that $\beta_c(\alpha)$ is periodic with period π. A feature of

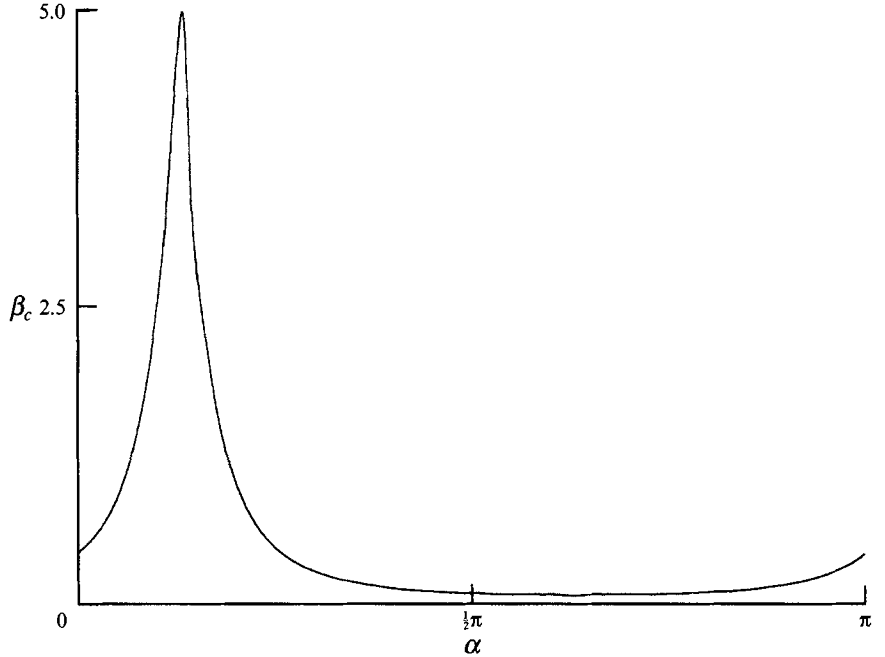

FIGURE 3. The critical value of $\beta = \beta_c(\alpha)$ for torsional oscillations of the boundary with

$$\phi(\tau) = \cos(\tau+\alpha), \lambda_2 = \beta\lambda_1.$$

For values of $\beta > \beta_c$ the flow becomes quasi-steady, whilst for $\beta < \beta_c$ there is a breakdown of the solution at a finite time.

figure 3 is that $\beta_c = 5$ when $\tan\alpha = \frac{3}{7}$, this corresponding to $g(0,\bar{\tau}) \equiv 0$. In turn, with $\beta = 5, f'(0,\bar{\tau}) = 0$ and in this region $f = g \equiv 0$, so that no outer boundary layer is needed in this exceptional case.

By comparing (37) and (44), it is worth noting that for the case of steady rotation a very much smaller value of λ_2 is necessary to inhibit the development of a singularity in the solution than when the boundary performs torsional oscillations.

(*b*) $\lambda_1 \gg 1$

Unlike the previous case, when $\lambda_1 \ll 1$, where lower bounds for λ_2, the amplitude of the azimuthal motion of the boundary, were made, the situation is not so straightforward for $\lambda_1 \gg 1$. Indeed a prohibitive amount of computing time would be required to obtain the same precise information. However it is still possible to demonstrate that a steady rotation, or azimuthal oscillatory motion, can suppress the eruptive, singular behaviour of the solution. We again consider each in turn.

(i) $\phi(\tau) \equiv 1, \tau > 0$. For this case we choose a sufficiently rapid rotation rate for the boundary by writing $\lambda_2 = \lambda_1^3$. An appropriate balance between the viscous and inertia terms in (8), (9) is then achieved by setting $\eta = \epsilon\zeta$, where $\epsilon = \lambda_1^{-1}$ so that, with a prime denoting differentiation with respect to ζ, we have

$$\tfrac{1}{2}F''' + 2FF'' - F'^2 + G^2 = O(\epsilon^2), \tag{50a}$$

$$\tfrac{1}{2}G'' + 2(FG' - F'G) = O(\epsilon^2), \tag{50b}$$

with $\quad F(0,\tau) = F'(0,\tau) = 0, \quad G(0,\tau) = 1, \quad F'(\infty,\tau) = O(\epsilon), \quad G(\infty,\tau) = 0. \quad (50c)$

The solution of these equations at leading order is that for a disk rotating steadily in a fluid otherwise at rest. So, by rotating the plane boundary sufficiently rapidly the centrifuge effect will overwhelm the tendency of the fluid to drift radially inwards and erupt at the stagnation point.

(ii) $\phi(\tau) = \cos\tilde{\tau}, \tau > 0$. In contrast to (*a*) (ii), the boundary performs torsional oscillations with a much higher frequency. Thus, $\phi = \cos\tilde{\tau}$ with $\tilde{\tau} = \lambda_1^4\tau$. We choose large-amplitude oscillations such that $\lambda_2 = \lambda_1^5$, and to effect a balance between the unsteady and viscous terms we set $\eta = \lambda_1^{-2}\zeta = \epsilon^2\zeta$. Equations (8), (9) and associated boundary conditions then become, with a prime denoting differentiation with respect to ζ,

$$\frac{\partial F'}{\partial\tilde{\tau}} + \epsilon(F'^2 - 2FF'' - G^2) - \tfrac{1}{2}F''' = O(\epsilon^5), \tag{51a}$$

$$\frac{\partial G}{\partial\tilde{\tau}} + 2\epsilon(F'G - FG') - \tfrac{1}{2}G'' = 0, \tag{51b}$$

$$F(0,\tilde{\tau}) = F'(0,\tilde{\tau}) = 0, \quad G(0,\tilde{\tau}) = \cos\tilde{\tau}, \qquad F'(\infty,\tilde{\tau}) = O(\epsilon^2), \quad G(\infty,\tilde{\tau}) = 0. \tag{51c}$$

Up to and including terms $O(\epsilon)$, equations (51) are as studied by Rosenblat (1959) and Riley (1965) in an investigation of the flow induced by the torsional oscillations of a disk in a fluid otherwise at rest. In particular, if we write

$$F(\zeta,\tilde{\tau}) = \sum_{n=0}^{\infty} \epsilon^n F_n(\zeta,\tilde{\tau}), \quad G(\zeta,\tilde{\tau}) = \sum_{n=0}^{\infty} \epsilon^n G_n(\zeta,\tilde{\tau}), \tag{52}$$

then we have $F_0 = G_1 \equiv 0$, and

$$G_0 = \mathrm{e}^{-\zeta}\cos(\tilde{\tau}-\zeta), \tag{53a}$$

$$F_1 = -\tfrac{1}{8}(1-2\zeta-\mathrm{e}^{-2\zeta}) - \tfrac{1}{16}\{(2-\surd 2)\cos(2\tilde{\tau}+\tfrac{1}{4}\pi) - 2\,\mathrm{e}^{-\surd 2\zeta}\cos(2\tilde{\tau}-\surd 2\zeta+\tfrac{1}{4}\pi+\surd 2\,\mathrm{e}^{-2\zeta}\cos(2\tilde{\tau}-2\zeta+\tfrac{1}{4}\pi)\}. \tag{53b}$$

From (53*b*) we have $F_1' \to \frac{1}{4}$ as $\zeta \to \infty$ which shows that a non-zero, time-averaged radial flow $O(\epsilon)$ persists at the edge of a layer of thickness $O\{\epsilon^2(\nu/\omega)^{\frac{1}{2}}\}$. To accommodate this in an outer region Riley (1965) writes $\tilde{\zeta} = \epsilon\zeta$, $F = \tilde{F}(\tilde{\zeta})$, $G = \tilde{G}(\tilde{\zeta})$, where it is shown that, at leading order, $\tilde{G} = 0$ and $\tilde{F}$ satisfies

$$\tfrac{1}{2}\tilde{F}''' + 2\tilde{F}\tilde{F}'' - \tilde{F}'^2 = 0, \tag{54a}$$

together with
$$\tilde{F}(0) = \tilde{F}'(\infty) = 0, \quad \tilde{F}'(\infty) = \tfrac{1}{4}, \tag{54b}$$

and a prime denotes differentiation with respect to $\tilde{\zeta}$. The solution of (54) is discussed by Riley (1965). In particular, we note for our purpose, that the time-independent radial outflow induced by the torsional oscillations of the boundary is of relative order ϵ^{-1} when compared with the flow associated with the free-stream oscillations. And so, again, we have a situation in which a net radial outflow inhibits any tendency for the flow to erupt from the stagnation point.

This brings us back to the problem of Amin & Riley (1990). Figure 1, as we have discussed in §1, shows the jet-like flow emerging from a sphere along the axis of oscillation in a quasi-steady manner, following an initial eruption of dye from the surface. It now seems clear that if the sphere were to perform some additional suitable torsional motion about the axis, there would be no net flow along the axis. Rather we might expect to see a net motion in a plane perpendicular to the axis in the form of a radial jet, as for example observed by Bowden & Lord (1963) in their experiments with a rotating sphere.

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